Kleine AG: Travaux de Shimura

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Synopsis

This Kleine AG grew from the wish to understand some aspects of Deligne's axiomatic definition of Shimura varieties and their canonical models, see [1]. The aims of this program are to illustrate the significance of such a definition for arithmetics by studying the modular curve (Talks 1 and 2), to motivate the general definition of Shimura variety by formulating the Siegel case (Talk 3), to give Deligne's definition (Talk 4) and to prove that the moduli description of the modular curve yields the canonical model in the case of GL_2 (Talk 5). We now describe these contents in more detail.

The modular curve

Consider the upper half plane $\mathfrak{h}$ together with its action by $\mathrm{GL}_2(\mathbb{R})_+$ through Moebius transformations. Recall that the principal congruence subgroup $\Gamma(N)$ for $N \geq 1$ is defined as

$$\Gamma(N) := \ker\big(\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\big).$$

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called congruence subgroup, if it contains $\Gamma(N)$ for some N. An important special case is

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod (N) \right\}.$$

For any congruence subgroup Γ, we define the modular curve of level Γ as the quotient

$$Y_\Gamma := \Gamma \setminus \mathfrak{h}.$$

It is naturally given the structure of a smooth affine complex algebraic curve and we denote its compactification by X_Γ. On $\mathfrak{h}$, there is a natural $\mathrm{GL}_2(\mathbb{R})_+$-equivariant line bundle ω, the so-called Hodge bundle. Γ-invariant sections of $\omega^{\otimes k}$ that also satisfy a certain growth condition at the boundary $X_\Gamma \setminus Y_\Gamma$ are called cuspidal modular forms of weight k and level Γ. We denote the space of such forms by $S_k(\Gamma)$. For example, $\omega^{\otimes 2} \cong \Omega^1_{\mathfrak{h}}$ and thus cuspidal modular forms of weight 2 are nothing but certain Γ-invariant differential forms. More precisely, the growth condition translates into an identification

$$S_2(\Gamma) = H^0(X_\Gamma, \Omega^1_{X_\Gamma}).$$

By Hodge decomposition, this implies

$$S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \cong H^1(X_\Gamma, \mathbb{C}). \tag{1}$$

Of course, $GL_2(\mathbb{R})_+$ does not act on X_Γ. Still, it is possible to define the family of so-called Hecke operators: Let $N \geq 1$ be minimal such that $\Gamma(N) \subseteq \Gamma$. For any $d \geq 1, (d, N) = 1$, there is a correspondence T_d on X_Γ, called Hecke correspondence, which acts on $S_k(\Gamma)$. The correspondence also acts on $H^1(X_\Gamma, \mathbb{C})$ and, by naturality, the isomorphism (1) is equivariant. Let $\mathbb{T}_k(\Gamma) \subseteq \mathrm{End}_{\mathbb{C}}(S_k(\Gamma))$ be the $\mathbb{Q}$-subalgebra generated by the T_d.

Theorem. *For $k, N \geq 1$, the ring $\mathbb{T}_k(\Gamma_0(N))$ is a finite-dimensional semi-simple commutative $\mathbb{Q}$-algebra and the representation $S_k(\Gamma_0(N))$ is defined over $\mathbb{Q}$.*

The starting point for the theory of canonical models is the observation that the curves Y_Γ, the descent of the bundles $\omega^{\otimes k}$ to Y_Γ and the Hecke correspondences T_d can all be defined algebraically over a number field. To explain ideas, we restrict to the case $k = 2$ and $\Gamma = \Gamma_0(N)$ from now on. Then $Y_0(N) := Y_{\Gamma_0(N)}$ is the coarse moduli space of pairs (E, C) of elliptic curves E together with a cyclic subgroup $C \subseteq E$ of order N. The Hecke correspondence T_d (restricted to the open $Y_0(N)$) is the coarse moduli of triples (E, C, D) where (E, C) is as above and $D \subseteq E$ is a subgroup of order d, with map

$$\begin{aligned} T_d|_{Y_0(N)\times Y_0(N)} &\longrightarrow Y_0(N) \times Y_0(N) \\ (E, C, D) &\longmapsto \big((E, C), (E/D, (C + D)/D)\big). \end{aligned}$$

Thus both $Y_0(N)$ and the T_d are defined over $\operatorname{Spec}\mathbb{Q}$. In particular, this shows that the representation of the Hecke algebra on $S_2(\Gamma_0(N)) = H^0(X_0(N), \Omega^1_{X_0(N)})$ is defined over $\mathbb{Q}$ which was one of the assertions of the above theorem.

We may also consider the $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $H^1_{\text{ét}}(X_0(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. By naturality, it carries a $\mathbb{T}_2(\Gamma_0(N))$-action for which both the Galois action and the comparison isomorphism

$$H^1(X_0(N)(\mathbb{C}), \mathbb{C}) \cong H^1_{\text{ét}}(X_0(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$$

are equivariant. Recall that the set of Frobenius elements F_p is dense in $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and thus their images determine any continuous Galois representation. For almost all primes p, $X_0(N)$ has good reduction at p and the action of F_p can be computed on the special fiber. Analyzing the moduli description given above yields the famous Eichler-Shimura congruence relation

$$T_p = F_p + F_p^*,$$

where F_p^* results from a certain involution applied to the Frobenius. It follows that F_p is a zero of

$$(U - F_p)(U - F_p^*) = U^2 - T_pU + p \in \mathbb{T}_2(\Gamma_0(N))[U].$$

In terms of eigenvalues, this implies that if a_p is an eigenvalue of T_p on $S_2(\Gamma_0(N))$ with multiplicity $n(a_p)$, then F_p has the two roots of $U^2 - a_pU + p$ as eigenvalues, each also with multiplicity $n(a_p)$. This characterizes the Galois representation attached to $X_0(N)$ in terms of the purely analytical object $S_2(\Gamma_0(N))$.

We may apply the above to the Hasse-Weil conjecture for $X_0(N)$. Recall that to any algebraic variety V over $\operatorname{Spec}\mathbb{Q}$ there is associated its Hasse-Weil ζ-function

$$\begin{aligned} \zeta^{(i)}(s, V/\mathbb{Q}) &:= \prod_p \det\left(1 - p^{-s}F_p | H^i_{\text{ét}}(V_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_{I_p}\right)^{-1}, \\ \zeta(s, V/\mathbb{Q}) &:= \prod_{i=0}^{2\dim(V)} \zeta^{(i)}(s, V/\mathbb{Q})^{(-1)^i} \end{aligned}$$

which is conjectured to extend meromorphically to the complex plane and to satisfy a functional equation. One strategy to prove this is to relate $\zeta(s, V/\mathbb{Q})$ to the L-function of an automorphic representation. The latter arises analytically, so its analytical properties are easier understood. In the case at hand, one can associate an L-function $L(s, f)$ to any $\mathbb{T}_2(\Gamma_0(N))$-eigenfunction $f \in S_2(\Gamma_0(N))$ and prove both the meromorphic continuation and the functional equation.

Theorem (Shimura [7]). *Let $f_1, \ldots, f_r \in S_2(\Gamma_0(N))$ be a basis consisting of $\mathbb{T}_2(\Gamma_0(N))$-eigenvectors. Then, up to possibly the Euler factors at the finitely many primes $p|N$,*

$$\zeta^{(1)}(s, X_0(N)/\mathbb{Q}) = \prod_{i=1}^{r} L(s, f_r).$$

In particular, $\zeta(s, X_0(N)/\mathbb{Q})$ extends meromorphically to the complex plane and satisfies a functional equation.

Underlying the above line of reasoning is an association of a Galois representation ρ_f to a Hecke eigenform $f \in S_2(\Gamma_0(N))$. Deligne extended this to weight $k > 2$, see [2]. These are nontrivial instances of the Langlands correspondence, which is conjectured to exist in much greater generality.

General Shimura Varieties

Variants of the modular curve have been studied for a long time. In general, the place of $\mathfrak{h}$ is taken by a hermitian symmetric domain or a finite union thereof X. It has a transitive action of a real Lie group $G(\mathbb{R})$ and there is an identification

$$X = G(\mathbb{R})/K_\infty$$

for a maximal connected subgroup $K_\infty \subseteq G(\mathbb{R})$ such that its image in the adjoint group $G(\mathbb{R})^{\mathrm{ad}}$ is compact. For any arithmetic subgroup $\Gamma \subseteq G(\mathbb{R})$, one can define both the quotient $X_\Gamma := \Gamma \setminus X$ and the notion of Γ-automorphic form. By Bailey-Borel, X_Γ is known to be a complex algebraic variety.

Example. If $X = \mathfrak{h}_g$ is the Siegel half space, then the group is $\mathrm{GSp}_{2g}(\mathbb{R})_+$. The arithmetic subgroups are those coming from symplectic lattices and the quotients $\Gamma\backslash\mathfrak{h}_g$ have an interpretation as coarse moduli spaces of polarized abelian varieties with level structure of a certain type.

Beginning from the late 1950s, Shimura studied in numerous cases the possibility of defining these varieties over number fields and its applications to arithmetic. The point being (in today's terminology) that the Galois representations arising in their étale cohomology can often be related to the automorphic forms side which leads to various cross-relations. Besides the Siegel case, Shimura studied general moduli problems of PEL-type and some non-PEL-type examples related to quaternion algebras over various fields.

In 1971, Deligne systematized the setup of Shimura and gave the purely group-theoretical definition of a Shimura variety and its canonical model over some specified number field, see [1]. He proves the uniqueness and, in case the group theoretic datum is of Hodge type, existence of such a model. It was later shown by Borovoi and Milne that canonical models exist in general.

Deligne also clarified the role played by the connected components of Shimura varieties. In the case of the modular curve sketched above, we had to restrict to level of the form $\Gamma_0(N)$ to get a model over $\mathbb{Q}$. For general Γ, the canonical model of X_Γ is defined over a finite extension of $\mathbb{Q}$ which depends on Γ. In Deligne's setup, $\mathfrak{h}$ and SL_2 are replaced by the non-connected $\mathbb{C} \setminus \mathbb{R}$ and GL_2. Then the whole tower of quotients is defined over $\mathrm{Spec}\,\mathbb{Q}$, with an explicit action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of geometric connected components.

Talks

The first two talks roughly follow the presentation in [8]. The main reference for the modular curve is still the comprehensive survey [3].

Talk 1 (**Analytic theory of the modular curve).** Recall the analytic definition of the modular curve and modular (cusp) forms for some level Γ, see [8, Section 2] and [3, Sections 2 and 7]. Introduce the ring of Hecke operators with its action on the space of cusp forms and interpret this by an action through correspondences, see [8, Sections 3 and 4] or [3, Section 3]. Finally, attach an L-function to a new form as in [3, Section 5].

Talk 2 (**Arithmetic theory of the modular curve).** For simplicity, restrict to the case of level $\Gamma = \Gamma_0(N)$. Follow [3, Section 8] to provide a model of the modular curve over $\mathbb{Q}$ by considering it as a moduli space of elliptic curves together with a subgroup of order N. Give a moduli theoretic interpretation of the Hecke correspondences T_p, thus also defined over $\mathbb{Q}$. Compute the reduction of T_p at p to prove the Eichler-Shimura congruence relation, see [3, Section 8.5]. Recall the definition of the Hasse-Weil ζ-function and prove that it is modular in the case at hand as in [8].

For the remaining three talks, we follow the presentation of Genestier and Ngô in [4], further information can be found in [2], [5] and [6].

Talk 3 (**Siegel upper half space).** The aim of this talk is to generalize both the group theoretical and the moduli description of the modular curves to higher dimensions, following [4, Section 1]. Recall the notion of a polarized abelian variety over $\mathbb{C}$, culminating in the description [4, Corollary 1.1.7]. Parameterize polarized abelian varieties by the Siegel half space and prove [4, Propositions 1.2.3 and 1.2.4]. (For simplicity, you may restrict to the principally polarized case.) Construct a tower of complex manifolds by adding level structure as in [4, Section 1.3]. Finally, state the theorem of existence of a moduli space of principally polarized abelian varieties over $\mathbb{Q}$, thus giving a model over a number field, see [4, Section 2.3] for the definition.

Talk 4 (**General Shimura Varieties).** Give the definition of a reductive Shimura datum and the important result [4, Proposition 4.3.2]. The latter also serves as a motivation for the notion of Shimura datum. Define the Shimura variety attached to a Shimura datum and state the results [4, Theorem 4.5.2 and Lemma 4.6.1] which show that this defines a tower of algebraic varieties. Cover the case $G = \mathrm{GL}_2$ in detail: Identify the set $\mathrm{GL}_2(\mathbb{A}_f) \times \mathfrak{h}^\pm$ with the set of isogeny classes of elliptic curves $E/\mathbb{C}$ together with choices of bases for both $H_1(E(\mathbb{C}), \mathbb{Z})$ and the full rational Tate-module $\mathbb{Q} \otimes_\mathbb{Z} \prod_p T_p(E)$. Deduce a description of the double quotient

$$\mathrm{GL}_2(\mathbb{Q}) \setminus \left[\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathfrak{h}^\pm\right]$$

as the set of elliptic curves with level structure of type K, up to isogeny. Recover the modular curves from Talk 1, at least up to connected components. See [4, Section 2.6] as a reference, but specialize to $g = 1$ in your talk.

Talk 5 (**Canonical models).** Define the canonical models of the Shimura varieties attached to Tori, see [4, Section 5.4]. Give the definition of the reflex field and the canonical model of a general Shimura variety. Prove that the moduli spaces of elliptic curves with level structure yield the canonical model for the modular curve (case of GL_2).

Literatur

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